

Beyond Lyapunov: Ergodic parameters and dynamical complexity

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Abstract

Ergodic parameters like the Lyapunov and the conditional exponents are global functions of the invariant measure, but the invariant measure itself contains more information. A more complete characterization of the dynamics by new families of ergodic parameters is discussed, as well as their relation to the dynamical Rényi entropies and measures of complexity. A generalization of the Pesin formula is derived which holds under weak correlation conditions.

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1 Introduction

Ergodic parameters associated to an invariant measure play a central role in the characterization of dynamical systems. In addition to rigorous notions of chaos, they also provide indicators of self-organization [1], sufficient conditions for self-organized criticality [2] and a characterization of topological transitions in networks [3].

The Lyapunov [4] and the conditional exponents [5] [6], are global functions of the invariant measure. However, the invariant measure itself contains

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more information. Ergodic parameters, being obtained from infinite-time limits, are averages of local fluctuating quantities. The quantity describing the fluctuations is again an ergodic parameter and the same reasoning applies in turn to its fluctuations, etc. [7]. Therefore, unless the fluctuations are fluctuations of a Gaussian random variable, to fully characterize the invariant measure, a much larger set of parameters is in general needed.

The task of constructing a larger set of ergodic parameters has already been addressed by several authors. For example, Farmer, Sidorowich and Dressler [8] [9] proposed to use infinite-time limits of higher order derivatives of the map. The existence status of these limits is weaker [10] (convergence in probability) than for the case of Lyapunov and conditional exponents. Also, whenever they exist, it turns out that they are simple functions of the Lyapunov exponents and therefore provide no new information. Other ergodic parameters, independent from the Lyapunov exponents, were proposed by several authors, either in the form of generalized entropies [11] [12] [13], as moments of the local fluctuations of the tangent vectors [14] [15] [11] [16] [17] [18] or from the eigenvalues of the Hessian in a variational formulation .

In this paper, a general cocycle formulation is used which allows to describe in a unified way the generalized ergodic parameters that have been proposed in the past as well as a new ergodic parameter that subsumes all the information on the statistics of local fluctuations of the expansion rate. That multi-point correlations should also be taken into account in the ergodic description of dynamical systems is pointed out.

Then, in Section 3, the moments of the local expansion rate are related to generalized entropies and in Section 4 one discusses how the ergodic parameters may be used to characterize and quantify the notions of complexity and dynamical self-organization.

2 Generalized ergodic parameters. A cocycle approach

Let $f : M \rightarrow M$ be a measure-preserving transformation of a Lebesgue space (M, \mathcal{B}, μ) . For any measurable function $g : M \rightarrow GL(N, \mathbb{R})$ and $x \in M$ define

$$C(x, n) = g(f^{n-1}(x)) \cdots g(x) \quad (1)$$

and $C(x, 0) = \text{Id}$. Then

$$C(x, n+k) = C(f^k(x), n) C(x, k) \quad (2)$$

and any measurable function $C : M \times \mathbb{Z} \rightarrow GL(N, \mathbb{R})$ satisfying (2) is called a *cocycle* (over f). Any cocycle has the form (1) and the map g is called the *generator* of the cocycle.

The Oseledets multiplicative ergodic theorem [20] is a powerful result insuring the existence of some infinite-time limits associated to a cocycle C . Namely, if

$$\ln_+ \|g(x)\| \in L^1(M, \mu) \quad (3)$$

then :

(i) there is a decomposition of \mathbb{R}^N

$$\mathbb{R}^N = \oplus_{i=1}^{k(x)} E_i(x) \quad (4)$$

invariant under $C(x, n)$,

(ii) and the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{\|C(x, n)v\|}{\|v\|} = \chi_i(x) \quad (5)$$

with

$$\chi_1(x) < \chi_2(x) < \dots < \chi_{k(x)}(x) \quad (6)$$

exist uniformly in $v \in E_i(x) \setminus \{0\}$.

If the generator of the cocycle is

$$g_1(x) = Df(x) = \exp(\ln(Df(x))) \quad (7)$$

the quantities $\chi_i(x)$ are the usual Lyapunov exponents of the dynamics f . If the full Jacobian Df is replaced by partial blocks of Df one obtains the conditional exponents [5] [6]. However, provided that the integrability condition (3) is satisfied, Oseledets' theorem applies to any other linear cocycle extension of f .

Definition 1 *The Lyapunov fluctuation moments $\chi_i^{(p)}(x)$ are defined as the limits (5) when the generator of the cocycle is*

$$g_p(x) = \exp(\ln_+^p(Df(x))) \quad (8)$$

The definition of the logarithm in (8) should be understood in the framework of the Oseledets-Pesin ε -reduction theorem [21] [22]. Namely, under the measurability conditions of the Oseledets theorem, for any $\varepsilon > 0$ there is an invertible map $\Gamma_\varepsilon(x) : M \rightarrow GL(N, \mathbb{R})$ such that the generator

$$g_\varepsilon(x) = \Gamma_\varepsilon^{-1}(f(x)) g(x) \Gamma_\varepsilon(x)$$

has block form, in each block

$$e^{\chi_i(x)-\varepsilon} \leq \|g_\varepsilon^i(x) v\| \leq e^{\chi_i(x)+\varepsilon}$$

and it generates a cocycle $C_\varepsilon(x, n)$ equivalent to $C(x, n)$. The \ln_+ in (8) is therefore computed without ambiguity in each block and one sees that the limit

$$\chi_i^{(p)}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{\|g_p(f^{n-1}(x)) \cdots g_p(x) v\|}{\|v\|}$$

is an ergodic average of the p -moment of the local (positive) expansion rate.

As a consequence of the Oseledets multiplicative ergodic theorem, Lyapunov fluctuation moments $\chi_i^{(p)}(x)$ exist whenever

$$\ln_+ \|g_p(x)\| \in L^1(M, \mu) \tag{9}$$

This cocycle construction provides a unified description of the fluctuation ergodic parameters previously considered by several authors [14] [11] [16] [17] [18].

Existence of the limit (9) depends on the integrability of

$$\exp \left(\sum k_i \lambda_i^p(x) \right)$$

$\lambda_i(x)$ being the local expansion rate at the point x and k_i the multiplicity of this particular rate. However, the expansion rate random variable may fail to have moments for large p . In that case complete characterization of the fluctuations may be obtained by the ergodic equivalent of the characteristic function.

Definition 2 *The Lyapunov characteristic fluctuation function $C(\alpha)$ is defined as the limit (5) when the generator of the cocycle is*

$$g_\alpha(x) = \exp(\exp(i\alpha \ln_+(Df(x)))) \tag{10}$$

As before, existence of $C(\alpha)$ depends on integrability of $\ln_+ \|g_\alpha(x)\|$ and, because $\exp(i\alpha \ln_+(Df(x)))$ is bounded, this is always fulfilled.

Although $C(\alpha)$ contains complete information on the statistical properties of the local fluctuation rate a full ergodic characterization of the dynamics should also contain information about correlations at different points. The ergodic parameters obtained from the Hessian in a variational formulation [19] already contain partial information on the correlations, but a full study of this problem is far from complete.

3 Dynamical Rényi entropies and fluctuations of the local expansion rate

Another way that has been used [11] [12] [13] [24] to go beyond the Lyapunov characterization is the construction of generalized entropies.

Let Φ be a partition of M and $\{\phi_i^{(n)}\}$ the elements of the partition Φ_n (partition refined by the dynamics f)

$$\Phi_n = \bigvee_{i=0}^{n-1} f^{-i}(\Phi) \quad (11)$$

Then, the dynamical Rényi entropy of order α is

$$K(\alpha) = \sup_{\Phi} \left\{ \lim_{n \rightarrow \infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \sum_i \mu(\phi_i^{(n)})^\alpha \right\} \quad (12)$$

The sup over all possible partitions (or the existence of a generating partition) is not, in general, easy to establish. Therefore, an easier to compute (but not necessarily equivalent) definition uses a partition of the phase-space in uniform boxes of side ε [23] [24]. Let the invariant measure be absolutely continuous with respect to Lebesgue. Then, denoting by $p(i_0 \cdots i_{n-1})$ the joint probability to be at the box i_0 at time 0, to be at box i_1 at time 1, \cdots , and to be at box i_{n-1} at time $n-1$

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \sum_{i_0 \cdots i_{n-1}} (p(i_0 \cdots i_{n-1}))^\alpha \quad (13)$$

the sum being over all different blocks of length n .

This is the definition that will be used here to obtain an estimate of its relation to the fluctuations of the local expansion rate. The local expansion rate $\Lambda(x) = \prod_{\lambda_i > 0} e^{\lambda_i(x)}$ of the dynamics (defined as in Section 2) implies that if the system is in box i_0 at time 0 it can go to $\Lambda(i_0)$ boxes in the next step, then to $\Lambda(i_0)\Lambda(i_1)$ boxes, etc. Here $\Lambda(i_k)$ denotes the average expansion rate in the (small) box i_k . Then, one obtains for the probability $p(i_0 \cdots i_{n-1})$ the following estimate [24]

$$p(i_0 \cdots i_{n-1}) = \frac{\mu(i_0)}{\Lambda(i_0) \cdots \Lambda(i_{n-2})} \quad (14)$$

$\mu(i_0)$ being the measure of the i_0 box. Hence

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{1 - \alpha} \frac{1}{n} \ln \left(q_n \left\langle \left(\frac{\mu(i_0)}{\Lambda(i_0) \cdots \Lambda(i_{n-2})} \right)^\alpha \right\rangle \right)$$

q_n being the number of different blocks of length n and $\langle \cdots \rangle$ denoting expectation values over blocks with this length. q_n is obtained from normalization $\sum_{i_0 \cdots i_{n-1}} p(i_0 \cdots i_{n-1}) = 1$. Then

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{1 - \alpha} \frac{1}{n} \ln \left(\left\langle \frac{\mu(i_0)}{\Lambda(i_0) \cdots \Lambda(i_{n-2})} \right\rangle^{-1} \left\langle \left(\frac{\mu(i_0)}{\Lambda(i_0) \cdots \Lambda(i_{n-2})} \right)^\alpha \right\rangle \right)$$

In the $\lim_{n \rightarrow \infty}$ one may write

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{1 - \alpha} \frac{1}{n} \ln \left\langle \exp \left((1 - \alpha) \sum_{k=0}^{n-2} \ln \Lambda(i_k) \right) \right\rangle \quad (15)$$

This establishes the relation between the dynamical Rényi entropy and what some authors [11] [24] [12] call *generalized Lyapunov exponents*.

One recognizes in the above expression $(1 - \alpha) K_B(\alpha)$ as the pressure function for the random variable $Y_n = \frac{1}{n} \sum_{k=0}^{n-2} \ln \Lambda(i_k)$ [27]. Therefore, if it is differentiable (in α), its Legendre transform

$$I(y) = \sup_{\alpha} \{ (1 - \alpha) y - (1 - \alpha) K_B(\alpha) \} \quad (16)$$

is the deviation function for the large deviations of the random variable $Y_n = \frac{1}{n} \sum_{k=0}^{n-2} \ln \Lambda(i_k)$, that is, it characterizes the probability P_n for finite-time fluctuations in the computation of the sum of the positive Lyapunov

exponents.

$$P_n \left\{ \frac{1}{n} \sum_{k=0}^{n-2} \ln \Lambda(i_k) \in (y, y + dy) \right\} \asymp \exp(-nI(y)) dy$$

the symbol \asymp meaning logarithmic equivalence.

This establishes a general relation between the dynamical Rényi entropy and the fluctuations of the local expansion rate. Under more strict conditions, that is, if the correlation between successive values of $\Lambda(i_k)$ decays sufficiently fast, namely if

$$\left\langle \exp \left((1 - \alpha) \sum_{k=0}^{n-2} \ln \Lambda(i_k) \right) \right\rangle \prod_{k=0}^{n-2} \langle \exp((1 - \alpha) \ln \Lambda(i_k)) \rangle^{-1} \leq c_1 e^{c_2 n^\gamma} \quad (17)$$

with $c_2 > 0$ and $\gamma < 1$, then

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \frac{1}{1 - \alpha} \ln \langle \exp((1 - \alpha) \ln \Lambda(i)) \rangle \quad (18)$$

which one recognizes as a cumulant generating function. Summarizing:

Proposition 1 *The Legendre transform of the (box) dynamical Rényi entropy is the deviation function of the local expansion rate. If the weak correlation condition (17) is verified then*

$$K_B(\alpha) = \lim_{\varepsilon \rightarrow 0} \sum_{s=1}^{\infty} k_s(\ln \Lambda) (1 - \alpha)^{s-1} \quad (19)$$

where $k_s(\ln \Lambda)$ are the cumulants of the local expansion rate.

In its range of validity Eq.(19) is a generalization of Pesin's formula [21]. Grassberger and Procaccia [26] have also obtained a similar, although more complex, relation between the dynamical Rényi entropy and the fluctuations of the expansion rate.

4 Ergodic parameters and measures of complexity

To have quantitative measures of complexity and self-organization is an important issue for a mathematical theory of complex systems. The *algorithmic*

complexity [28] [29] of the signal generated by a dynamical system, that is, of the sequence of numbers coding a particular orbit, is the limit

$$C_K(S) = \lim_{n \rightarrow \infty} \frac{M_n(S)}{n} \quad (20)$$

where $M_n(S)$ is the length of the smallest program (code plus data) able to generate the first n symbols of the sequence. Up to a factor, the average algorithmic complexity of the sequences is identical to the *Shannon entropy* [30] of the system considered as a source emitting the sequence.

The notion of algorithmic complexity applies to each particular sequence, whereas an ergodic invariant like the *Kolmogorov-Sinai entropy* is a statistical parameter referring to the average behavior of the orbits. Nevertheless, the two notions are related. Let in $M_n(S)$ distinguish two components

$$M_n(S) = c_1(n) + c_2n \quad (21)$$

where $c_1(n)$ is the length of the code and c_2n the size of the input data. c_2n is the part of the information that is not explained by the program code. Therefore, as far as the model program is concerned, c_2n is the random component of the sequence. In general $\frac{c_1(n)}{n} \rightarrow 0$ when $n \rightarrow \infty$ and only the random component contributes to the algorithmic complexity. For this reason, in many cases, the algorithmic complexity of typical orbits coincides (up to a factor) with the Kolmogorov-Sinai entropy [31] [32].

The algorithmic complexity, the Shannon entropy and the Kolmogorov-Sinai entropy (rate) measure the degree of unpredictability (or irregularity) of the system but not necessarily the difficulty of modelling it from experimental observations. In fact a system generating completely random sequences has maximum algorithm complexity, but may be modelled by a simple random number generator.

A better characterization of what is usually meant by complexity is the notion of *excess entropy* [33] or *effective measure complexity* [13] [23]. Let $p_N(s_1 \cdots s_n)$ be the probability to find the block $s_1 \cdots s_n$ of size n . Then

$$H(n) = - \sum_{\{s_i\}} p_n(s_1 \cdots s_n) \log p_n(s_1 \cdots s_n) \quad (22)$$

and

$$h_s = \lim_{n \rightarrow \infty} \frac{1}{n} H(n) \quad (23)$$

is the *Shannon entropy*.

The difference $\frac{1}{n}H(n) - h_s$ represents the additional information (beyond the one obtained from size n blocks) that is needed to reveal the true long-term unpredictability of the system. Summing all these differences, the *excess entropy* E grows with the amount of effort (and time) that is needed to construct an accurate model of the system.

$$E = \sum_n \left(\frac{1}{n}H(n) - h_s \right) \quad (24)$$

It is a measure of the diversity of dynamical structures that is present in the information source. The nature of the information processing employed by the dynamical system to produce its unpredictability is captured by the *statistical complexity* C_s [36] [37], related to the excess entropy by

$$E \leq C_s \quad (25)$$

meaning that, given an event, the ideal prediction of another one requires an amount of information at least equal to the mutual information between the two events.

The Kolmogorov-Sinai entropy, bounded by the sum of the positive Lyapunov exponents (an infinite-time average), is a measure of the complexity of typical orbits. On the other hand, it is to be expected that the finite-time fluctuations in the calculation of the Lyapunov exponents be a symptom of the diversity of dynamical structures. Therefore these fluctuations might be related to the excess entropy and therefore be a measure of the dynamical complexity of the system. Here such a relation is established.

One uses the large deviation principle that states that the Legendre transform $I(y)$ (Eq.(16)) of $(1 - \alpha) K_B(\alpha)$ (in Eq.(15)) is the deviation function of the random variable $Y_n = \frac{1}{n} \sum_{k=0}^{n-2} \ln \Lambda(i_k)$. For invariant measures absolutely continuous with respect to Lebesgue, the average value of Y_n is an estimate of $\frac{1}{n}H(n)$. Therefore one may write

$$E_e = \sum_n \left\{ \int_0^\infty y P_n(y) dy - y_{I_{\min}} \right\} \quad (26)$$

with $y_{I_{\min}}$ being the value that minimizes $I(y)$ and

$$P_n(y) = \frac{e^{-nI(y)}}{\int_0^\infty e^{-nI(y)} dy} \quad (27)$$

One sees that a *dynamical complexity measure* E_e analogous to the excess entropy E may be computed from the ergodic parameters that define the fluctuations of the local expansion rate.

In Ref.[40], the authors have proposed to measure the increase of *self-organization* between time t_1 and t_2 by the change in the statistical complexity

$$\Delta C_s = C_s(t_2) - C_s(t_1) \quad (28)$$

if this is not due to the action of an external agent. Given the relation (25) this might also be estimated by the change of excess entropy.

This might be an appropriate notion when one is comparing two different states of an evolving system. There is however another aspect of what is usually understood as self-organization in multi-agent systems that relates to the interrelation between the dynamics of the agents (and their local cluster) and the global collective dynamics. This aspect is better characterized by the relation between the Lyapunov exponents and the conditional ones (see [6] and [1] for details).

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